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Some Connection Theorem between Modified Laplace Transform and Modified Fractional Integral

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Abstract: The present paper establish some connection theorem between the modified Laplace transform and modified Fractional Integral. Some special cases and a relevant corollary of our main results are also proven.

Keywords - *Modified Laplace transform, Modified Fractional Integral, Mellin Transform*

1. Introduction :

Many Mathematicians take a keen interest in the field of fractional calculus[7]. Also, many of the authors[5],[6],[10], etc., give a fairly good account of the development and consider several aspects of applications to potential problems in analysis.[2],[4] etc. e.g. To solve the fractional differential equations Fractional, the fractional sub-equation method was proposed by[1]. General and but very important theorem which interconnects the Laplace transform and the generalized Weyl fractional integral operator involving the multivariable H-function of related functions of several variables is given by[9], A theorem which obtained the image of modified H-transform under the fractional integral operator involving Foxs H-function by [4],[7], The applications of fractional in several fields of engineering and science like viscoelasticity, fluid mechanics, electro-chemistry, biological population models, optics, and signals processing[9]. etc.,

The modified Laplace Transform $L^0(p) = \Lambda[f]$ of the function $f: R_+^n \to C$ is defined as

$$L^0(p) = \Lambda[f](p) = \int_{R_+^n} \exp[-\max(p_1 t_1, \dots \dots . p_n t_n)]\, f(t) dt \qquad \dots .(1.1)$$

The set of all points $p \in R^n$ such that the integral in[1.1] converges.

The second modified Laplace Transform $L_0(p) = v[f]$ of a $f: R_+^n \to C$ is defined as

$$L_0(p) = v[f]\,(p) = \int_{R_+^n} \exp[-\min(p_1 t_1, \dots \dots . p_n t_n)]\, f(t) dt \qquad ...(1.2)$$

The set of all points $p \in R^n$ such that the integral in [1.2] converges.

The modified Fractional Integrals[7] for $Re(\alpha) > 0$. The multidimensional modified integrals of order $\alpha \in C$ of $f: R_+^n \to C$ is defined by

$$S_{+;n}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_{1.......}\partial x_{n.}} \int_{R_+^n} [min\left(\frac{x_1}{t_1}, \dots \dots \frac{x_n}{t_n}\right) - 1]_+^{\alpha} . {}_2F_1\Big[\alpha+\beta, \alpha+\eta; 1+\alpha; 1 - min\left(\frac{x_1}{t_1}, \dots \dots \frac{x_n}{t_n}\right)\Big] f(t) dt$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [\, x_k \int_0^1 t^{n-\alpha-1} (1-t)^{\alpha} . {}_2F_1\Big[\alpha+\beta, \alpha+\eta; 1+\alpha; 1 - \frac{1}{t}\Big] f(x_1 t, \dots . . x_n t) dt.$$

……(1.3)

And

$$S_{-;n}^{\alpha,\beta,\eta} f(x) = \frac{(-1)^n}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_{1.......}\partial x_{n.}} \int_{R_+^n} [1 - max\left(\frac{x_1}{t_1}, \dots \dots \frac{x_n}{t_n}\right) - 1]_+^{\alpha} . {}_2F_1\Big[\alpha+\beta, -\eta; 1+\alpha; 1 - max\left(\frac{x_1}{t_1}, \dots \dots \frac{x_n}{t_n}\right)\Big] f(t) dt$$

$$= \frac{-1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [\, x_k \int_1^{\infty} t^{n-\alpha-1} (t-1)^{\alpha} . {}_2F_1\Big[\alpha+\beta, -\eta; 1+\alpha; 1 - \frac{1}{t}\Big] f(x_1 t, \dots . . x_n t) dt. \qquad (1.4)$$

To prove the main result we need the following lemmas.

Lemma 1([11, P.153 Theorem 3]): Let $f \in M_\gamma(R_+^n)$; $g \in M_{1-Re(d)-\gamma}(R_+^n)$ and $|d| = 1 - \eta - \beta$. Then

$$\int_{R_+^n} x^{-d} g(x) S_{-;n}^{\alpha,\beta,\eta} f(x) dx = \int_{R_+^n} x^{-d} f(x) S_{-;n}^{\alpha,\beta,\eta} g(x) dx \qquad ….(1.5)$$

Provided that $Re(\alpha) > 0; \gamma_j + Re(d_j) < 0\ (j = 1, \dots \dots, n);$

$|\gamma| < \eta\ + \min((Re(\beta), Re(\eta);\ |\gamma| + Re(|d|) < 1 - Re(\beta - \eta)$.

Lemma 2: If $s = (s_1 \dots \dots., s_n) \in C^n, h = (h_1 \dots \dots., h_n) \in R_+^n, p = (\boldsymbol{p_1} \dots \dots., \boldsymbol{p_n}) \in R_+^n$

And $g^{\left|\frac{s}{h}\right|-1} g(y) \in L_1(R_+)$, then

(a) $\int_{R_+^n} x^{s-1} g(\max(p_1 x_1^{h_1}, \dots, p_n x_n^{h_n}))\, dx = \frac{|\frac{s}{h}|}{\boldsymbol{s_1.,s_n(}} (|p|^{-|s|}) g^* \left(\left|\frac{s}{h}\right|\right),$

$Re(s_j) < 0\ (j = 1,2,3 \dots, n)$ ….(1.6)

[b]$\int_{R_+^n} x^{s-1} g(\min(p_1 x_1^{h_1} \dots \dots p_n x_n^{h_n}))\, dx = \frac{(-1)^{n-1}|\frac{s}{h}|}{\boldsymbol{s_1 \dots\dots, s_n(}} (|p|^{-|s|}) g^* \left(\left|\frac{s}{h}\right|\right),$

$Re(s_j) < 0\ (j = 1,2,3 \dots, n)$ ….(1.7)

Where g^* denotes the one-dimensional Mellin transform of $g(u)$. We give outlines of the proof of the result. We have

LHS of (1.6) $= \int_{R_+^n} x^{s-1} g(\max(p_1 x_1^{h_1} \dots \dots p_n x_n^{h_n}))\, dx$

$$= \sum_{k=1}^{n} \int_0^\infty {x_k}^{s_k - 1} g(p_k x_k^{h_k}) \{ \int_0^{x_k} \underset{\dots.}{\overset{k}{v}} \int_0^{x_k} \frac{x^{s-1}}{x_k^{s_k-1}} dx_1 \underset{\dots.}{\overset{k}{v}} dx_k \} dx_k.$$

$$= \sum_{k=1}^{n} \int_0^\infty {x_k}^{s_k - 1} g(p_k x_k^{h_k}) \{ \prod_{j=1; j \neq k}^{n} \frac{x_k^{s_j}}{s_j} \} dx_k.$$

$$\sum_{k=1}^{\infty} \frac{s_k}{s_1 \dots \dots s_n..} \frac{1}{h_k} (\frac{1}{p_k})^{|s|} \int_0^\infty x_k^{\frac{|s|}{h_k}-1} g(x_k)\, dx_k.$$

= RHS of (1.6)

The second assertion of (1.7) of the lemma can be proved similarly.

2. Main Result

Connection Theorem1:Let $s \in R_+^n$, and the modified Laplace transform of the operator

$S_{+;n}^{\alpha,\beta,\eta}$ and $S_{-;n}^{\alpha,\beta,\eta}$ exist, then

$$L^0\left\{S_{+;n}^{\alpha,\beta,\eta} f\right\}(s) = \int_{R_+^n} E_1(n,\alpha,\beta,\eta,s,x)\, f(x)dx \qquad \ldots(2.1)$$

And

$$L_0\left\{S_{-;n}^{\alpha,\beta,\eta} f\right\}(s) = \int_{R_+^n} E_2(n,\alpha,\beta,\eta,s,x)\, f(x)dx \quad \ldots Re(s) > 0 \qquad \ldots..(2.2)$$

Where.

$$E_1(n,\alpha,\beta,\eta,s,x) = S_{-;n}^{\alpha,\beta,\eta}\{e^{-\max(s_1x_1\ldots\ldots\ldots s_nx_n)}\}$$

$$= G_{2.2}^{3,0}[\max(s_1x_1 \ldots\ldots\ldots s_nx_n)|\begin{matrix}(1-\beta-n,1),\ (1+\alpha+\eta-n,1)\\(1-n,1)(1-\beta+\eta-n,1),(0,1)\end{matrix} \qquad \ldots(2.3)$$

And

$$E_2(n,\alpha,\beta,\eta,s,x) = S_{+;n}^{\alpha,\beta,\eta}\{e^{-\min(s_1x_1\ldots\ldots\ldots s_nx_n)}\}$$

$$= G_{2.3}^{1,2}[\min(s_1x_1 \ldots\ldots\ldots s_nx_n)|\begin{matrix}(1-\beta-n,1),\ (1+\alpha+\eta-n,1)\\(0,1),(1-n,1)(1-\alpha-\beta+\eta-n),\end{matrix}$$

$$= \frac{\Gamma(n+\beta)\Gamma(n+\eta)}{\Gamma n\Gamma(n+\alpha+\beta+\eta)}\,{}_2F_1\left[\begin{matrix}n+\beta, n+\eta\\n, n+\alpha+\beta+\eta\end{matrix}; -min(s_1x_1 \ldots\ldots\ldots s_nx_n)\right] \qquad \ldots.(2.4)$$

Proof: By definition (1.1) we have

$$L^0\left\{S_{+;n}^{\alpha,\beta,\eta} f\right\}(s) = \int_{R_+^n} S_{-;n}^{\alpha,\beta,\eta} \{e^{-\max(s_1x_1 \ldots\ldots s_nx_n)}\{ S_{+;n}^{\alpha,\beta,\eta} f\}dx$$

$$= \int_{R_+^n} f(x)\{S_{-;n}^{\alpha,\beta,\eta} e^{-\max(s_1x_1 \ldots\ldots s_nx_n)}\}dx$$

$$= \int_{R_+^n} f(x)E_1(n,\alpha,\beta,\eta,s,x)dx \qquad \ldots(2.5)$$

On using lemma 2 in (2.5) we obtain the result (2.1), the second assertion (2.2) of theorem 1 can be proved similarly.

Theorem2: Let the modified fractional integrals of the modified Laplace transform of a function, exist then

$$S_{+;n}^{\alpha,\beta,\eta}\{L_0[f](s)\} = \int_{R_+^n} f(x)E_2(n,\alpha,\beta,\eta,s,x)\, f(x)dx \qquad \ldots.(2.6)$$

And

$$S_{+;n}^{\alpha,\beta,\eta}\{L^0[f](s)\} = \int_{R_+^n} f(x)E_1(n,\alpha,\beta,\eta,s,x)\, f(x)dx \qquad (Re(s) > 0) \quad \ldots..(2.7)$$

And where $E_1(n,\alpha,\beta,\eta,x,s)$ and $E_2(n,\alpha,\beta,\eta,x,s)$ are given by [2.3] and [2.4] with x and s interchanging in it.

Proof: Proceeding on lines similar to theorem 1, we can easily establish theorem 2.

3. Special cases.

If we take $\beta = -\alpha$ in theorem 1 and 2 we get the following result:

Corollary1 : $L^0\{X_+^\alpha f\}(s) = \int_{R_+^n} G_{1.2}^{2,0}\left[\max(s_1x_1 \ldots\ldots\ldots s_nx_n)\left|\begin{matrix}(1+\alpha-n,1),\ (1+\alpha-n,1)\\(1-n,1),(0,1)\end{matrix}\right.\right] f(x)dx$

(3.1)

for $(Re(s) > 0; (Re(\alpha) > 0, |\gamma| > n-1$ and

$$L_0\{\{X_-^\alpha f\}(s)\} = \frac{\Gamma(n-\alpha)}{\Gamma n)} \int_{R_+^n} 1F1[n-\alpha; n; -\min(s_1x_1 \ldots \ldots \ldots s_nx_n)]f(x)dx \qquad \ldots\ldots(3.2)$$

for $(Re(s) > 0; (Re(\alpha) > 0, (Re(\alpha) > |\gamma| > n-1$

Corollary2

$$X_+^\alpha \{L_0[f](s)\} = \frac{\Gamma(n-\alpha)}{\Gamma n)} \int_{R_+^n} 1F1[n-\alpha; n; -\min(s_1x_1 \ldots \ldots \ldots s_nx_n)]f(x)dx \qquad \ldots(3.3)$$

for $(Re(s) > 0; (Re(\alpha) > 0, (Re(\alpha) > |\gamma| > n-1$ and

$$X_-^\alpha \{L^0[f]f(s)\} = \int_{R_+^n} G_{1.2}^{2,0}\left[\max(s_1x_1 \ldots \ldots \ldots s_nx_n)\left|\begin{matrix}(1+\alpha-n,1),\ (1+\alpha-n,1)\\ (1-n,1),(0,1)\end{matrix}\right.\right]f(x)dx$$

..(3.4)

For $Re(s) > 0; (Re(\alpha) > 0, (Re(\alpha) > |\gamma| > n-1$ where X_+^α and X_-^α denote the modified fractional integral operator defined by[2] etc. Which are, of course also special cases of the operators (1.3) and(1.4). For $\beta = -\alpha$

Conclusion: Fractional Integral and Laplace transform both are very important branches of mathematics. In the paper, we have established the connection theorem between the Modified Laplace Transform and Modified Fractional Integral.

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